AFM Notes, Unit 1 Probability

1-7 Binomial Theorem

Name ____________________

Date 2-11-2020 Period 2

According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. Find these sequences.

GBGB BGBG GGBB BBGG BGGB GBBG

Pascal's Triangle

You can use the coefficients in powers of binomials to count the number of paossible sequences in situation such as the one above. Remember that a binomial is a polynomial with two terms. Expand a few powers of the binomial b + g.

$(b+g)^0 = 1b^0g^0 = 1$

$(b+g)^1 = 1b^1g^0 + 1b^0g^1 = 1b + 1g$

$(b+g)^2 = 1b^2g^0 + 2b^1g^1 + 1b^0g^2 = 1b^2 + 2bg + 1g^2$

$(b+g)(b+g)$

$(b+g)^3 = 1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3 = 1b^3 + 3b^2g + 3bg^2 + 1g^3$

$(b+g)(b+g)(b+g)$

$(b+g)^4 = 1b^4g^0 + 4b^3g^1 + 6g^2b^2 + 4b^1g^3 + 1b^0g^4 = 1b^4 + 4b^3g + 6b^2g^2 + 4bg^3 + 1g^4$

The Coefficient 6 of the b^2g^2 in the expansion of $(b + g)^4$ gives the number of sequences of births that result in two boys and two girls. As another example the coefficient 4 of the b^1g^3 term gives the number of sequences with one boy and three girls.

Here are some patterns that can be seen in any binomial expansion of the form $(a + b)^n$.

1. There are n+1 terms.
2. The exponent n of $(a+b)^n$ is the exponent of *a* in the first term and the exponent of *b* in the last term.
3. In successive terms, the exponent of *a* decreases by one, and the exponent of *b* increases by one.
4. The sum of the exponents in each term is n.
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

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The coefficients form a pattern that is often displayed in a triangular formation. This is known as Pascal's Triangle. Notice that each row begins and ends with 1.
Each coefficient is the sum of the two coefficients about t in the previous row

1
1 1
1 2 1

$(a+b)^0$	1						
$(a+b)^1$	1	1					
$(a+b)^2$	1	2	1				
$(a+b)^3$	1	3	3	1			
$(a+b)^4$	1	4	6	4	1		
$(a+b)^5$	1	5	10	10	5	1	
$(a+b)^6$	1	6	15	20	15	6	1

Example 1: Use Pascal's Triangle
Expand $(x + y)^7$
Write two more rows of Pascal's triangle.

1 7 21 35 35 21 7 1

Use the patterns of a binomial expression and the coefficients to write the expansion of $(x+ y)^7$

$(x + y)^7 =$

$$1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7$$

$$\boxed{x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7}$$

Notes 1-7 Binomial Theorem

The Binomial Theorem

Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

(a + b)0 $\quad 1$

(a + b)1 $\quad 1 \quad \frac{1}{1}$

(a + b)2 $\quad 1 \quad \frac{2}{1} \quad \frac{2\cdot1}{1\cdot2}$

(a + b)3 $\quad 1 \quad \frac{3}{1} \quad \frac{3\cdot2}{1\cdot2} \quad \frac{3\cdot2\cdot1}{1\cdot2\cdot3}$

(a + b)4 $\quad 1 \quad \frac{4}{1} \quad \frac{4\cdot3}{1\cdot2} \quad \frac{4\cdot3\cdot2}{1\cdot2\cdot3} \quad \frac{4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4}$

This pattern provides the coefficients of (a + b)n for any nonnegative integer n. The pattern is summarized in the Binomial theorem.

If n is a nonnegative integer, then

(a + b)n = $1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1\cdot2}a^{n-2}b^2 + \dots + 1a^0b^n$

Example 2: Use the Binomial Theorem

Expand $(a - b)^6$

The expansion will have 7 terms. Use the sequence $1, \frac{6}{1}, \frac{6\cdot5}{1\cdot2}, \frac{6\cdot5\cdot4}{1\cdot2\cdot3}$ to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

(a - b)6 = $1a^6b^0 - \frac{6}{1}a^5b^1 + \frac{6\cdot5}{1\cdot2}a^4b^2 - \frac{6\cdot5\cdot4}{1\cdot2\cdot3}a^3b^3 + \dots 1a^0b^6$

$a^6 - 6a^5b^1 + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

Note:

The factors in the coefficients of binomial expansion involve special products called factorials. For example, the product of $4\cdot3\cdot2\cdot1$ is written 4! And is read as 4 factorial. In general, if n is positive integer, the $n! = n(n-1)(n-2)(n-3)\ldots$.

Example 3: Factorials

Evaluate $\frac{8!}{3!5!}$

$$\frac{8\cdot7\cdot6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{3\cdot2\cdot1\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 56$$

An expression such as $\frac{6\cdot5\cdot4}{1\cdot2\cdot3}$ in Example 2 can be written as a quotient of factorials. In this case $\frac{6!}{3!\,3!}$. Using this idea, you can rewrite the expansion of (a + b)6 using factorials:

$$(a+b)^6 = \frac{6!}{6!\cdot0!}a^6b^0 + \frac{6!}{5!\cdot1!}a^5b^1 + \frac{6!}{4!\,2!}a^4b^2 + \frac{6!}{3!\cdot3!}a^3b^3 + \frac{6!}{2!\cdot4!}a^2b^4 + \frac{6!}{1!\,5!}a^1b^5 + \frac{6!}{0!\cdot6!}a^0b^6$$

You can also write this series using sigma notation.

$$(a+b)^6 = \sum_{k=0}^{6} \frac{6!}{(6-k)!\,k!}\cdot a^{6-k}b^k$$

In general. the Binomial theorem can be written both in factorial noation and in sigma notation.

Example 4: Use a Factorial Form of the Binomial Theorem $(2x+y)^5$

$$\sum_{k=0}^{5} \frac{5!}{(5-k)!\,k!} \cdot (2x)^{5-k} y^k$$

$$\frac{5!}{5!\,0!} \cdot (2x)^{5-0} y^0 + \frac{5!}{4!\,1!} \cdot (2x)^{5-1} y^1 + \frac{5!}{3!\,2!} \cdot (2x)^{5-2} y^2 + \frac{5!}{2!\,3!} (2x)^{5-3} y^3 + \frac{5!}{1!\,4!} \cdot (2x)^{5-4} y^4 + \frac{5!}{0!\,5!} (2x)^{5-5} y^5$$

$$32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$$

Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, k = 0 for the first time, k = 1 for the second term, and so on. In general, the value of k is always one less than the number of term you are finding.

Example 5: Find a Particular Term

Find the fifth term in the expansion of (p + q)10.

First, use the Binomial Theorem to write the expansion in sigma notation.

$$\sum_{k=0}^{10} \frac{10!}{(10-k)!\,k!} p^{10-k} q^k$$

If the fifth term, k = 4.

$$\frac{10!}{((10-4)!\cdot 4!)} p^{10-4} q^4$$

$$210\, p^6 q^4$$